NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 2

Assessment Task 2 Term 1 2014

Name: __________________ **Mathematics Class:** 12MZ__

Student Number: ________________

Time Allowed: 55 minutes + 2 minutes reading time

Available Marks: 39

Instructions:

- Questions are *not* of equal value.
- Start each question in a new booklet.
- Show all necessary working.
- Do not work in columns.
- Marks may be deducted for incomplete or poorly arranged work.

Question	1-3	4-5	6a-bii	6biii-c	7a	7b	Total
E4	/3		/8				/11
E6		/2				/13	/15
E9				/8	/5		/13
							/39

Section 1

Multiple Choice Section **5 marks**

Answer all the questions for the multiple choice on the answer sheet provided for Section I.
Choose the best response that is correct for the question.
Marks will not be deducted for incorrect answers.
No marks will be given if more than one answer is provided for any question.

1. If $x^3+px^2+qx+r=(x-\alpha)(x-\beta)^2$, which statement is NOT correct?

(A) $\alpha^3+p\alpha^2+q\alpha+r=0$ (B) $3\alpha^2+2p\alpha+q=0$

(C) $\beta^3+p\beta^2+q\beta+r=0$ (D) $3\beta^2+2p\beta+q=0$

2. The polynomial equation $x^3=2-3x$ has roots α, β and γ.
What is the value of $\alpha^3+\beta^3+\gamma^3$?

(A) 2 (B) 6

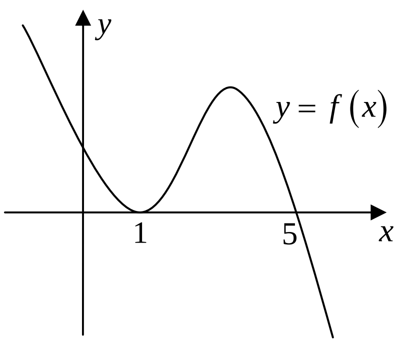
(C) 11 (D) 15

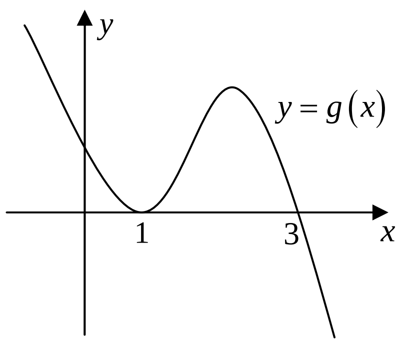
3. $P(z)=(x+i)Q(z)$ where $P(z)$ is a polynomial with only real coefficients.
Which of the following statements is not necessarily true?

(A) $Q(i)=0$ (B) $Q(-i)=0$

(C) $Q(z)$ has at least one complex coefficient (D) $P(i)=0$

4. Consider the following graphs of $y=f(x)$ and $y=g(x)$:



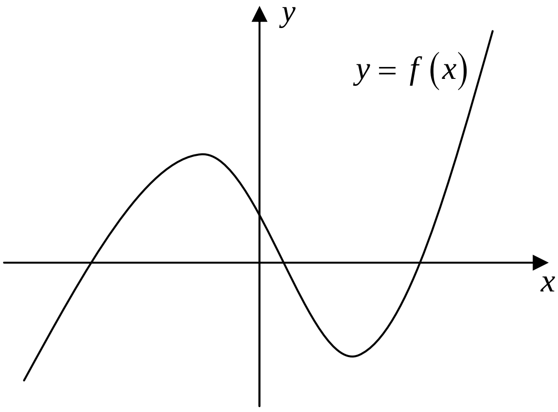


Which of the following statements could be true?

(A) $g(x)=f(2x-1)$ (B) $g(x)=f[2(x-1)]$

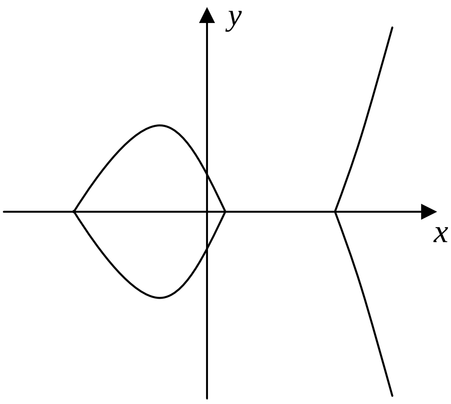
(C) $g(x)=f\left(\dfrac{x-1}{2}\right)$ (D) $g(x)=f\left(\dfrac{x}{2}-1\right)$

5. Consider the function $y = f\left(x\right)$ as shown.

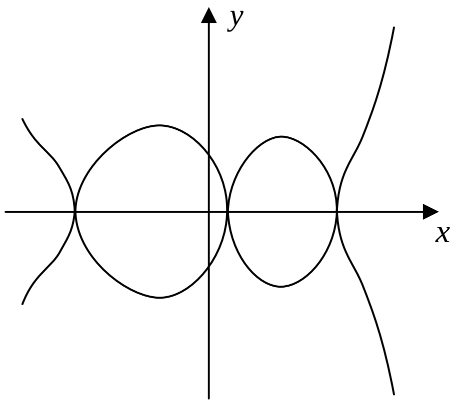


Which of the following could be the graph of $y^2 = f\left(x\right)$?

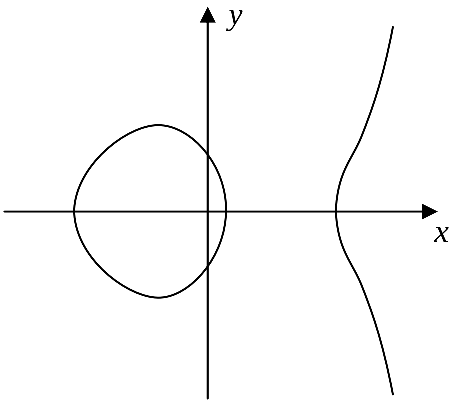
(A)



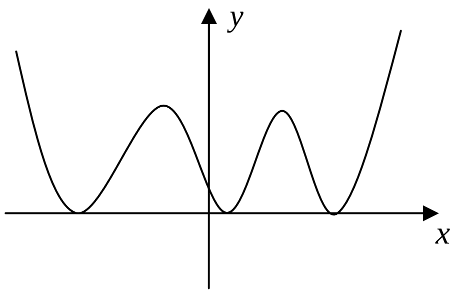
(B)



(C)



(D)



Section 2 – Free Response

Question 6 (Answer in a new writing booklet) **(16 Marks)**

(a) Consider the polynomial $P(x) = x^4 - 2x^3 + 6x^2 - 8x + 8$.

(i) Given that $1+i$ is a root of $P(x) = 0$, state why $1-i$ is also a root. **1**

(ii) Write $P(x)$ as the product of two quadratic expressions. **2**

(iii) Fully factorise $P(x)$ over the complex numbers. **1**

(b) You are given that the roots of the equation $x^3 + px^2 + qx + r = 0$ are α, β and γ.

(i) Show that the polynomial equation with roots α^2, β^2 and γ^2 is **3**

$$x^3 + \left(2q - p^2\right)x^2 + \left(q^2 - 2pr\right)x - r^2 = 0.$$

(ii) Hence or otherwise find an expression for $\alpha^2 + \beta^2 + \gamma^2$. **1**

(iii) **<u>Hence</u>** explain why the equation $x^3 + 3x^2 + 5x + k = 0$ (where k is real) has exactly one real root, regardless of the value of k. (Don't differentiate) **2**

(c) (i) Find the five complex solutions of $z^5 = 1$, writing your answers in modulus-argument form with arguments θ satisfying $-\pi \le \theta \le \pi$. **2**

(ii) Hence show that $z^5 - 1 = (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$. **2**

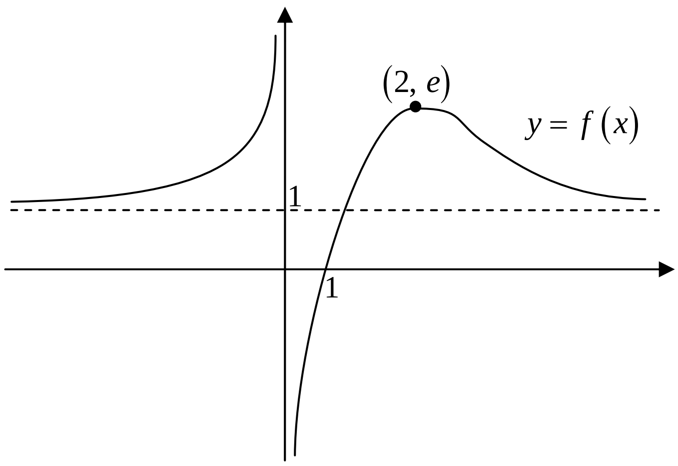
(iii) By factorizing $z^5 - 1$ another way, or otherwise, show that $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$. **2**

Question 7 (Start a new answer booklet)

(a) (i) Prove by mathematical induction that $(1+x)^n - 1$ is divisible by x for integers $n \geq 1$. **3**

(ii) Hence or otherwise prove that $35^n - 7^n - 5^n + 1$ is divisible by 24 for integers $n \geq 1$. **2**

(b) Consider the following graph of the function $y = f(x)$:



On separate number planes, sketch graphs of

(i) $y = f(|x|)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \log_e f(x)$ **2**

(c) Consider the relation defined implicitly by $(x-y)^2 = 4(x+y-1)$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{x-y-2}{x-y+2}$. **2**

(ii) Show that the only stationary point on the curve occurs at $(2,\ 0)$. **2**

(Substitution of the coordinates into the derivative alone will earn no marks.)

(iii) By referring to part (b) or otherwise, state why the only vertical tangent is at $(0, 2)$. **1**

(iv) The original equation may be rewritten in the form $(x-1)^2 + (y-1)^2 = \left(\dfrac{x+y}{\sqrt{1^2+1^2}}\right)^2$. **3**

(**DO NOT** SHOW THIS.)

Use this rearranged form to justify why the curve is a parabola, and write down the coordinates of the focus and equation of the directrix.

Hence sketch the curve.

12. The equation $z^n = 2^n$ has roots $2,\ z_1,\ z_2,\ z_3,\ \ldots,\ z_n$. **2**

Show that $(2-z_1)(2-z_2)(2-z_3)\ldots(2-z_n) = n\left(2^{n-1}\right)$.

Extension 2 Assessment Task 2 2014 Suggested Solutions

Section 1 – Multiple Choice

1. B

2. B

3. B

4. A

5. C

Working:

1. Let $P(x) = x^3 + px^2 + qx + r$.

Since α is a root of $P(x) = 0$, both $P(\alpha) = 0$ [A]

and $P(\beta) = 0$ [C].

Since β is a double root of $P(x) = 0$ then $P'(\beta) = 0$ [D] $\quad \therefore$ [B]

2. Since α, β and γ are roots:

$$\alpha^3 = 2 - 3\alpha$$
$$\beta^3 = 2 - 3\beta$$
$$\gamma^3 = 2 - 3\gamma$$

Adding:
$$\begin{aligned}\alpha^3 + \beta^3 + \gamma^3 &= 6 - (\alpha + \beta + \gamma)\\ &= 6 - 0\\ &= 6\end{aligned}$$

[B]

3. Since $z = -i$ is a zero of $P(z)$ and $P(z)$ is real, then $z = i$ is also a zero.

$\therefore$ $(z - i)$ must be a factor of $Q(z)$

This information tells us that $Q(i) = 0$ [A] and $P(i) = 0$ [D].

It is possible that $(z + i)$ is also a factor of $Q(z)$, but there is nothing to guarantee this.

Since $P(x)$ is real, ALL instances of the factor $(z + i)$ must be paired with a factor of $(z - i)$.

But since $Q(z)$ is missing one of those factors of $(z + i)$, it must have a factor of $(z - i)$ that is NOT paired with a factor of $(z - i)$. So it cannot be a real polynomial.

[C] must be true. $\quad \therefore$ [B]

4. From the graphs: $\quad g(3) = f(5) \quad [=0]$ and $g(1) = f(1) \quad [=0]$

The only option which gives both these results is $g(x) = f(2x - 1)$:

$$\left[\begin{array}{l}\text{replacing } x \text{ by 3 gives } g(3) = f[2(3) - 1] = f(5)\\ \text{replacing } x \text{ by 1 gives } g(1) = f[2(1) - 1] = f(1)\end{array}\right]$$

[A]

5.
- If $y^2 = f(x)$ then $f(x) \ge 0$, so drop the parts of the graph where $f(x) < 0$
- The curve becomes vertical at all single roots
- $y = \pm\sqrt{f(x)}$, so there is symmetry about the x-axis.

[C]

Section 2 – Free Response

Question 6

(a)

Consider the polynomial $P(x) = x^4 - 2x^3 + 6x^2 - 8x + 8$.

(i)

Given that $1+i$ is a root of $P(x) = 0$, state why $1-i$ is also a root. **1**

$P(x)$ has only real coefficients, and non-real roots of real polynomials occur in conjugate pairs.

(ii)

Write $P(x)$ as the product of two quadratic expressions. **2**

Let roots be $1+i,\ 1-i,\ \alpha,\ \beta$.

Sum: $2+\alpha+\beta = 2$
$\alpha+\beta = 0$

Product: $(1-i)(1+i)\alpha\beta = 8$
$2\alpha\beta = 8$
$\alpha\beta = 4$

$\therefore$ quadratic with roots $\alpha,\ \beta$ is x^2+4

also, quadratic with roots $1+i,\ 1-i$ is $x^2 - 2\,\mathrm{Re}(1+i)x + |1+i|^2 = 0$

$$x^2 - 2x + 2 = 0$$

$\therefore\ P(x) = (x^2-2x+2)(x^2+4)$

(iii)

Fully factorise $P(x)$ over the complex numbers. **1**

$$P(x) = [x-(1+i)][x-(1-i)][x-2i][x+2i]$$

(b)

You are given that the roots of the equation $x^3 + px^2 + qx + r = 0$ are α, β and γ.

(i)

Show that the polynomial equation with roots α^2, β^2 and γ^2 is **3**

$$x^3 + \left(2q - p^2\right)x^2 + \left(q^2 - 2pr\right)x - r^2 = 0.$$

Method 1

Let $P(x) = x^3 + px^2 + qx + r$

Equation with roots $\alpha^2, \beta^2, \gamma^2$ is $P\left(\sqrt{x}\right) = 0$

$$\begin{aligned}
\left(\sqrt{x}\right)^3 + p\left(\sqrt{x}\right)^2 + q\left(\sqrt{x}\right) + r &= 0 \\
x\sqrt{x} + px + q\sqrt{x} + r &= 0 \\
(x+q)\sqrt{x} &= -(px+r) \\
(x+q)^2 \cdot x &= (px+r)^2 \\
x^3 + 2qx^2 + q^2x &= p^2x^2 + 2prx + r^2 \\
x^3 + \left(2q - p^2\right)x^2 + \left(q^2 - 2pr\right)x - r^2 &= 0
\end{aligned}$$

Method 2

Equation with roots $\alpha^2, \beta^2, \gamma^2$ is $x^3 - \left(\alpha^2 + \beta^2 + \gamma^2\right)x^2 + \left(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2\right)x - \alpha^2\beta^2\gamma^2 = 0$

$$\begin{aligned}
\alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) \\
&= p^2 - 2q
\end{aligned}$$

$$\begin{aligned}
\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 &= (\alpha\beta)^2 + (\alpha\gamma)^2 + (\beta\gamma)^2 \\
&= (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2(\alpha\beta \cdot \alpha\gamma + \alpha\beta \cdot \beta\gamma + \alpha\gamma \cdot \beta\gamma) \\
&= (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \\
&= q^2 - 2(-r)(-p) \\
&= q^2 - 2pr
\end{aligned}$$

$$\begin{aligned}
\alpha^2\beta^2\gamma^2 &= (\alpha\beta\gamma)^2 \\
&= r^2
\end{aligned}$$

$\therefore$ equation is $x^3 + \left(2q - p^2\right)x^2 + \left(q^2 - 2pr\right)x - r^2 = 0$

(ii)

Hence or otherwise find an expression for $\alpha^2 + \beta^2 + \gamma^2$. **1**

$\alpha^2 + \beta^2 + \gamma^2 = -$coefficient of $x^2 = p^2 - 2q$

(iii)

Hence explain why the equation $x^3+3x^2+5x+k=0$ (where k is real) has exactly one real root, regardless of the value of k. (Don't differentiate) **2**

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2&=(3)^2-2(5)\\&=-1\\&<0\end{aligned}$$

The only way for the sum of squares to be negative is if at least one of these roots is non-real.

But the polynomial has only real coefficients, so it must have a second non-real root (the complex conjugate of the first root).

The third root must be real as it has no roots left to form its conjuage.

$\therefore$ this polynomial has exactly one real root, regardless of the value of k.

(c) (i)

Find the five complex solutions of $z^5=1$, writing your answers in modulus-argument form with arguments θ satisfying $-\pi\le\theta\le\pi$. **2**

$$\begin{aligned}z^5&=1\\&=\operatorname{cis}(2k\pi)\\z&=\operatorname{cis}\left(\frac{2k}{5}\pi\right)\qquad\text{(de Moivre's theorem)}\end{aligned}$$

Choosing $k=-2,-1,0,1,2$: $\quad z=\operatorname{cis}\left(-\frac{4\pi}{5}\right),\ \operatorname{cis}\left(-\frac{2\pi}{5}\right),\ \operatorname{cis}0=1,\ \operatorname{cis}\frac{2\pi}{5},\ \operatorname{cis}\frac{4\pi}{5}$

(ii)

Hence show that $z^5-1=(z-1)\left(z^2-2z\cos\frac{2\pi}{5}+1\right)\left(z^2-2z\cos\frac{4\pi}{5}+1\right)$. **2**

$$\begin{aligned}z^5-1&=(z-1)\left[z-\operatorname{cis}\frac{2\pi}{5}\right]\left[z-\operatorname{cis}\left(-\frac{2\pi}{5}\right)\right]\left[z-\operatorname{cis}\frac{4\pi}{5}\right]\left[z-\operatorname{cis}\left(-\frac{4\pi}{5}\right)\right]\\&=(z-1)\left[z^2-2\operatorname{Re}\left(\operatorname{cis}\frac{2\pi}{5}\right)z+\left|\operatorname{cis}\frac{2\pi}{5}\right|^2\right]\left[z^2-2\operatorname{Re}\left(\operatorname{cis}\frac{4\pi}{5}\right)z+\left|\operatorname{cis}\frac{4\pi}{5}\right|^2\right]\\&=(z-1)\left(z^2-2z\cos\frac{2\pi}{5}+1\right)\left(z^2-2z\cos\frac{4\pi}{5}+1\right)\end{aligned}$$

(iii)

By factorizing $z^5 - 1$ another way, or otherwise, show that $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$. **2**

$$z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$$

$$\therefore\ z^4 + z^3 + z^2 + z + 1 = \left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$$

Equating coefficients of z:

$$\begin{aligned} 1 &= -2\cos\frac{2\pi}{5} - 2\cos\frac{4\pi}{5} \\ &= -2\cos\frac{2\pi}{5} + 2\cos\frac{\pi}{5} \end{aligned}$$

$$(\div 2)\quad \frac{1}{2} = \cos\frac{\pi}{5} - \cos\frac{2\pi}{5}$$

Question 7

(a) (i)

Prove by mathematical induction that $(1+x)^n - 1$ is divisible by x for integers $n \geq 1$. **3**

Test $n=1$: $(1+x)^1 - 1 = 1 + x - 1$

$$= x$$

$$= x(1) \quad \text{(where 1 is a polynomial, not an integer)}$$

Assume true for $n=k$: ie. $(1+x)^k - 1 = x \cdot P(x)$ where $P(x)$ is a polynomial

Prove true for $n=k+1$: ie. RTP $(1+x)^{k+1} - 1 = x \cdot Q(x)$ where $Q(x)$ is a polynomial

$$\begin{aligned} \text{LHS} &= (1+x)^{k+1} - 1 \\ &= (1+x)(1+x)^k - 1 \\ &= (1+x)\left[x \cdot P(x) + 1\right] - 1 \\ &= x \cdot P(x) + 1 + x^2 \cdot P(x) + x - 1 \\ &= x \cdot P(x) + x^2 \cdot P(x) + x \\ &= x\left[(1+x)P(x) + 1\right] \end{aligned}$$

Since $P(x)$ is a polynomial, so is $(1+x)P(x)$, hence also $(1+x)P(x)+1$

$\therefore$ true for $n=k+1$ when true for $n=k$

$\therefore$ By mathematical induction, $(1+x)^n - 1$ is divisible by x for $n = 1, 2, 3, \ldots$

(ii)

Hence or otherwise prove that $35^n - 7^n - 5^n + 1$ is divisible by 24 for integers $n \geq 1$. **2**

$$\begin{aligned} 35^n - 7^n - 5^n + 1 &= 7^n \cdot 5^n - 7^n - 5^n + 1 \\ &= 7^n\left(5^n - 1\right) - 1\left(5^n - 1\right) \\ &= \left(5^n - 1\right)\left(7^n - 1\right) \end{aligned}$$

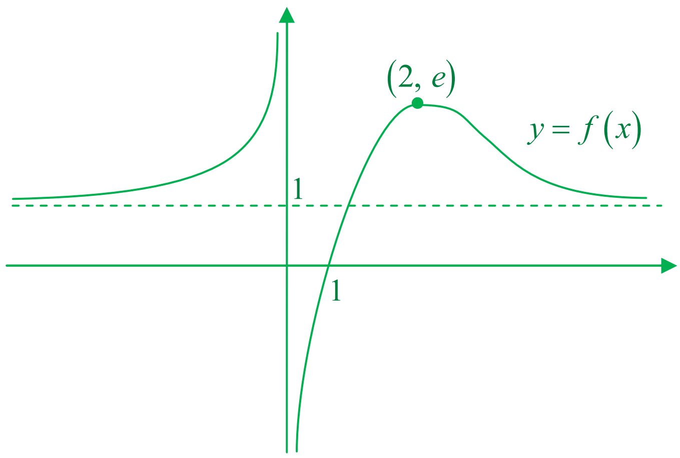
From part (i): $5^n - 1 = (1+4)^n - 1$ which is divisible by 4

and $7^n - 1 = (1+6)^n - 1$ which is divisible by 6

$\therefore$ $35^n - 7^n - 5^n + 1$ is divisible by $4 \times 6 = 24$

(b)

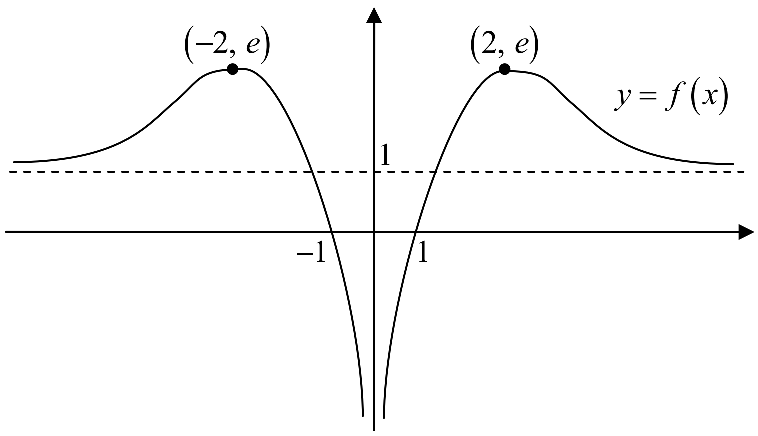
Consider the following graph of the function $y = f(x)$:



On separate number planes, sketch graphs of

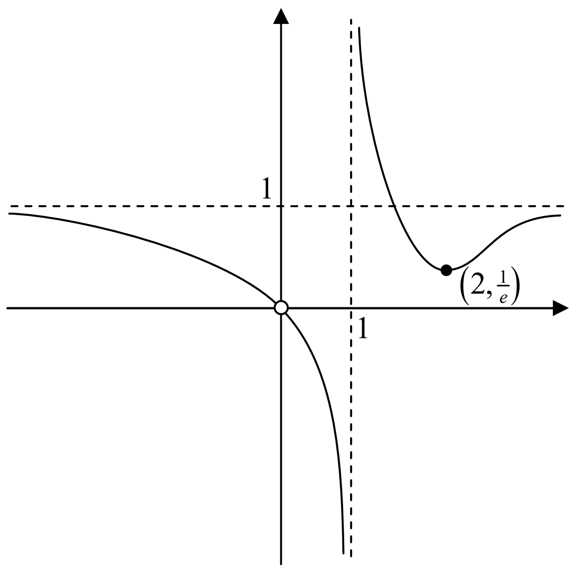
(i)

$y = f(|x|)$ **1**



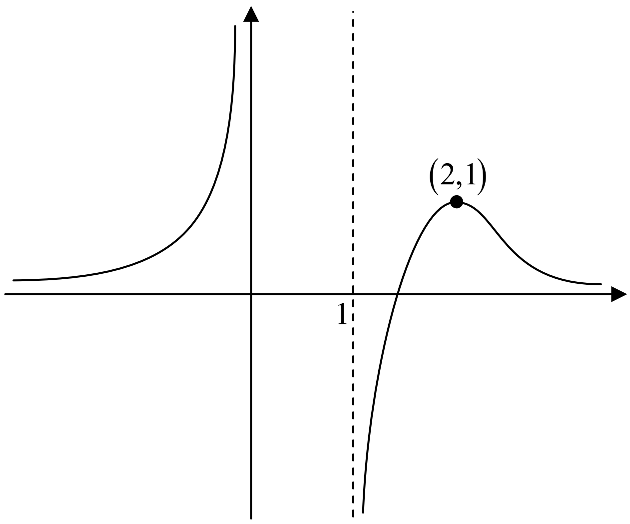
(ii)

$y = \dfrac{1}{f(x)}$ **2**



(iii)

$y = \log_e f(x)$ **2**



(c)

Consider the relation defined implicitly by $(x-y)^2 = 4(x+y-1)$.

(i)

Show that $\frac{dy}{dx} = \frac{x-y-2}{x-y+2}$. **2**

$$(x-y)^2 = 4(x+y-1)$$
$$2(x-y)\left(1-\frac{dy}{dx}\right) = 4\left(1+\frac{dy}{dx}\right)$$
$$2(x-y)-2(x-y)\frac{dy}{dx} = 4+4\frac{dy}{dx}$$
$$(\div 2) \qquad (x-y)\frac{dy}{dx}+2\frac{dy}{dx} = (x-y)-2$$
$$\frac{dy}{dx} = \frac{x-y-2}{x-y+2}$$

(ii)

Show that the only horizontal point on the curve occurs at $(2, 0)$. **2**
(Substitution of the coordinates into the derivative alone will earn no marks.)

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x-y-2=0 \quad \Rightarrow \quad y = x-2$$

Substitute into original equation: $\left[x-(x-2)\right]^2 = 4\left[x+(x-2)-1\right]$

$$2^2 = 4(2x-3)$$
$$2x-3 = 1$$
$$x = 2$$
$$y = 2-2 = 0$$

$\therefore$ only horizontal point at $(2,0)$

(iii)

By referring to part (b) or otherwise, state why the only vertical point is at $(0,2)$. **1**

Algebraicly swapping the roles of x and y leads to the same equation.

Graphically swapping the roles of x and y leads to a reflection in the line $y = x$, so the curve must be symmetrical in the line $y = x$.

So if the curve has a horizontal point at $(2,0)$ then it must have a vertical point at $(0,2)$.

(iv)

The original equation may be rewritten in the form $(x-1)^2 + (y-1)^2 = \left(\frac{x+y}{\sqrt{1^2+1^2}}\right)^2$. **3**

(**DO NOT** SHOW THIS.)

Use the rearranged form to justify why the curve is a parabola, and write down the coordinates of the focus and equation of the directrix.

Hence sketch the curve.

The LHS of the rearranged form is the square of the distance of a point (x, y) from $(1,1)$.

The RHS is the square of the perpendicular distance of (x, y) from the line $x + y = 0$.

Since these distances (squared) are equated, the resulting formula is the locus of all points equidistant from $(1,1)$ and $x + y = 0$.

That is, it is the parabola with focus $(1,1)$ and directrix $y = -x$.